

The University of Chicago

MULTIPLE INTEGRAL PROBLEMS
IN PARAMETRIC FORM IN
THE CALCULUS OF VARIATIONS

A DISSERTATION SUBMITTED TO THE FACULTY OF THE
DIVISION OF THE PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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By J. ERNEST WILKINS, JR.

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1. Introduction

It is the purpose of this paper to give a discussion of the simplest problem for multiple integrals in parametric form of the calculus of variations, namely, the problem of minimizing a multiple integral

$$(1:1) \quad I(V) = \int \cdots \int f(y^1, \cdots, y^n, p_1^1, \cdots, p_m^n) dt_1 \cdots dt_m,$$

where $p_a^i = \partial y^i / \partial t_a (i = 1, \cdots, n; a = 1, \cdots, m < n)$, in a class of varieties represented parametrically by equations of the form $y^i = y^i(t)$, all of which have the same boundary. In order that the integral (1:1) be independent of the parametric representation of the variety, it is known [XI] that $f(y, p)$ must satisfy the homogeneity condition

$$(1:2) \quad f(y, pB) = |B| f(y, p),$$

where B is an arbitrary m -rowed square matrix with positive determinant $|B|$, and pB is the matrix product of the matrices p and B .

In Section 3 we develop some of the elementary consequences of the homogeneity condition (1:2). For the most part these consequences are direct generalizations of results known to be true for $m = 1$. In Section 5 we give additional consequences of the identity (1:2) which involve the derivatives of $f(y, p)$ of arbitrary order. In order to describe these results in a reasonably compact form, we find it desirable to employ certain notational conventions which we explain in Section 4. In case $n = m + 1$ certain additional consequences of the homogeneity condition are deduced in Section 6.

After formulating the problem of the calculus of variations in Section 7, we proceed to discuss the necessary conditions for a minimum. With the exception of the Jacobi condition, these necessary conditions are immediate consequences of the corresponding necessary conditions for non-parametric problems. For the conditions of Weierstrass and Legendre, no satisfactory result is known for non-parametric problems and consequently a similar situation exists for parametric problems. One apparent objection is that it is not true that the ordinary E -function $E_0(y, p, P)$ must be non-negative for all choices of the matrix $P \neq p$, but only if $P - p$ has rank one. For an example, see Bliss [I, p. 80]. Carathéodory [V] has proposed an E -function $E_c(y, p, P)$ which he found useful in

¹ This paper is a revision of a doctor's dissertation accepted by the University of Chicago. The author is indebted to Professor M. R. Hestenes for many helpful comments.

making sufficiency proofs. We give in Section 9 an example to show that $E_c \geq 0$ is not necessary for a minimum. By analyzing the properties which an E -function should possess, we arrive at a new E -function $E_1(y, p, P)$ which possesses all of these properties and which reduces to E_0 in case $P - p$ has rank one. Moreover, this function vanishes identically for integrands of invariant integrals, although E_0 and E_c do not possess this property. On the basis of these results for the Weierstrass condition, we state the analogous results for the Legendre condition. We state and prove a Jacobi condition in Section 11.

Because of the hiatus in the field theory for non-parametric problems, we are not able to obtain a complete field theory for our parametric problem. We can, however, define a field in Section 12 and state conditions necessary and sufficient that an integral (1:1) be invariant in Section 13.

2. Integrals independent of the parametric representation

Suppose that we are given an integral of the form

$$(2:1) \quad I(V) = \int_T f(t, y, p) dT,$$

where $t = (t_1, \dots, t_m)$, $y = (y^1, \dots, y^n)$, $p = (p_a^i) = (\partial y^i / \partial t_a)$ ($i = 1, \dots, n$; $a = 1, \dots, m < n$), $dT = dt_1 dt_2 \dots dt_m$. The variety or manifold V will be represented by functions of the form $y^i = y^i(t)$ for t in some region T such that the matrix p has rank m on T . When these functions are of class C' , i.e., when they have continuous first partial derivatives in T , the integral (2:1) is well-defined. For a parametric problem, however, we must require that the integral (2:1) have the same value for any parametric representation of the same variety V , regarded as being oriented in a fixed manner. The condition for this was investigated by Lewis [XI] and we state it as

THEOREM 2:1. *The integral (2:1) is independent of the parametric representation if and only if the integrand function is a function $f(y, p)$ independent of t and such that for every m -rowed square matrix B with positive determinant $|B|$ the homogeneity condition*

$$(2:2) \quad f(y, pB) = |B| f(y, p)$$

holds. In this equation pB is the matrix product of p and B .

The integrand function $f(y, p)$ can also be regarded as a function of y and the m -rowed minors of the matrix p as has been shown by Grosz [VIII]. This representation will be of little use to us unless $n = m + 1$ in which case his result may be stated as

THEOREM 2:2. *In case $n = m + 1$ define the m -rowed minors A_i of p to be the cofactors of x_i in the matrix $(p_a^i x_i)$. Then the integral (2:1) is independent of the parametric representation if and only if the integrand is a function $G(y, A)$ which is positively homogeneous of the first degree in the variables A_i .*

Since the matrix p has rank m it follows easily that there exists a matrix $q = (q_j^i)$ ($j = 1, \dots, n - m$) of rank $n - m$ such that

$$(2:3) \quad p'q = 0, \quad |p'p| = |pq|,$$

where the prime denotes transposition. The matrix q is of course not uniquely determined by p since qC will satisfy (2:3) for any $(n - m)$ -rowed square matrix C with determinant unity. Moreover, if q and q^* are determined by p and p^* , then because of the second condition in (2:3) $q^* = qC$ if and only if $p^* = pB$ with $|C| = |B|$. With these comments in mind it is easy to prove that the integrand of an integral (2:1) which is independent of the parametric representation may also be regarded as a function $H(y, q)$ which satisfies the homogeneity condition

$$(2:4) \quad H(y, qC) = |C| H(y, q)$$

for every $(n - m)$ -rowed square matrix C with positive determinant. This criterion is not especially valuable to us unless $n = m + 1$, in which case it is equivalent to that supplied by Theorem 2:2. It does, however, display a type of duality in our problem in which $n - m$ and m are interchanged.

3. First consequences of the homogeneity condition

During the remainder of this paper we shall assume that $f(y, p)$ is a function satisfying the homogeneity condition (2:2) which possesses as many derivatives as are needed in any particular discussion. In this section certain elementary consequences of equation (2:2) will be deduced. Most of these results may be found in the paper of Lewis [XI] already referred to, but our proof of the fundamental relation (3:1) is considerably simpler.

THEOREM 3:1. *Let $f_i, f_i^a, f_{ik}^a, f_{ik}^{ab}$ ($a, b = 1, \dots, m; i, k = 1, \dots, n$) be the derivatives*

$$f_i = \partial f / \partial y^i, \quad f_i^a = \partial f / \partial p_a^i, \quad f_{ik}^a = \partial f_k / \partial p_a^i, \quad f_{ik}^{ab} = \partial f_i^a / \partial p_b^k.$$

*Then, with the notation C_a^b for the cofactor of B_b^a in the matrix B , the following identities are satisfied:*²

$$(3:1) \quad p_a^i f_i^b = \delta_a^b f;$$

$$(3:2) \quad f_i(y, pB) = |B| f_i(y, p);$$

$$(3:3) \quad p_a^i f_{ik}^b = \delta_a^b f_k;$$

$$(3:4) \quad B_b^a f_i^b(y, pB) = |B| f_i^a(y, p);$$

$$(3:5) \quad f_i^b(y, pB) = C_a^b f_i^a(y, p);$$

$$(3:6) \quad p_a^i f_{ik}^{bc} = \delta_a^b f_k^c - \delta_a^c f_k^b = (\delta_a^b \delta_d^c - \delta_a^c \delta_d^b) f_k^d;$$

$$(3:7) \quad p_a^i (f_{ik}^{bc} + f_{ik}^{cb}) = 0;$$

$$(3:8) \quad p_a^i p_d^k f_{ik}^{bc} = (\delta_a^b \delta_d^c - \delta_a^c \delta_d^b) f.$$

² We employ the usual convention concerning summation over repeated indices.

In all of these formulas the symbol δ_a^b is the Kronecker delta which has the value one for $b = a$ and the value zero otherwise. We obtain (3:1) by differentiating (2:2) with respect to B_b^a and setting B equal to the identity matrix. The other identities are easily obtained from (2:2) and this one by appropriate differentiations.

THEOREM 3:2. *There exist functions R_{jl}^{ab} , C_{jl}^{ab} ($a, b = 1, \dots, m; j, l = 1, \dots, n - m$) such that*

$$(3:9) \quad R_{jl}^{ab} = R_{jl}^{ba} = R_{li}^{ab}, \quad C_{jl}^{ab} = -C_{jl}^{ba} = -C_{li}^{ab},$$

$$(3:10) \quad f_{ik}^{ab} + f_{ik}^{ba} = 2 R_{il}^{ab} q_l^i q_i^k,$$

$$(3:11) \quad f(f_{ik}^{ab} - f_{ik}^{ba}) - 2(f_{ij}^{ab} - f_{ij}^{ba}) = 2 C_{jl}^{ab} q_j^i q_i^k,$$

where the q_j^i were defined in Section 2.

By (3:7) there exist quantities A_{kj}^{ab} such that $f_{ik}^{ab} + f_{ik}^{ba} = A_{kj}^{ab} q_j^i = A_{ij}^{ab} q_i^k$. Thus $(A_{ij}^{ab} p_c^i) q_j^k = 0$, $A_{ij}^{ab} p_c^i = 0$, so that there exist quantities R_{jl}^{ab} such that $A_{ij}^{ab} = 2 R_{jl}^{ab} q_l^i$, and then (3:10) holds. The symmetric relations expressed by the first equations in (3:9) are then easy consequences of the fact that the matrix q has rank $n - m$. To prove the second part of the theorem, we find by using (3:1) and (3:6) that

$$p_c^i [f(f_{ik}^{ab} - f_{ik}^{ba}) - 2(f_{ij}^{ab} - f_{ij}^{ba})] = 0.$$

Then an argument precisely the same as that used in proving the first part of the theorem suffices to complete the proof.

The Euler expressions $T_i(V)$ for a variety V admitting a representation $y^i = y^i(t)$ of class C'' are defined as

$$(3:12) \quad T_i(V) \equiv f_i - \partial f_i / \partial t_b,$$

where the arguments of the derivatives of f are those belonging to V . Let V be a fixed variety and define

$$(3:13) \quad 2\omega(t, \eta, \pi) \equiv f_{ik} \eta^i \eta^k + 2f_{ki}^c \eta^i \pi_c^k + f_{ik}^{bc} \pi_b^i \pi_c^k \quad (f_{ik} = \partial f_i / \partial y_k).$$

Then the Jacobi expressions $J_i(\eta)$ are defined by

$$(3:14) \quad J_i(\eta) \equiv \omega_i - \partial \omega_i / \partial t_b,$$

where $\omega_i = \partial \omega / \partial \eta^i$, $\omega_i^b = \partial \omega / \partial \pi_b^i$, $\pi_b^i = \partial \eta^i / \partial t_b$.

THEOREM 3:3. *If V is a variety of class C'' , the following identities subsist between its Euler and Jacobi expressions:*

$$(3:15) \quad T_i(V) p_a^i = 0$$

$$(3:16) \quad J_i(\eta) p_a^i = -T_i(V) \pi_a^i.$$

Moreover, if V is of class C''' , then for every set of functions $Z^a(t)$ of class C'' the functions $\phi^i = p_a^i Z^a$ are of class C'' and we have

$$(3:17) \quad J_i(\phi) = \partial(Z^a T_i) / \partial t_a.$$

Using (3:1) equation (3:15) follows from the relation

$$\begin{aligned} T_i(V)p_a^i &= f_i p_a^i - \partial(p_a^i f_i^b)/\partial t_b + p_{ab}^i f_i^b \\ &= \partial f/\partial t_a - \partial(\delta_a^b f)/\partial t_b = 0, \end{aligned}$$

where $p_{ab}^i = \partial p_a^i/\partial t_b = p_{ba}^i$. Replacing in (3:15) the functions $y^i(t)$ defining V by $y^i(t) + e\eta^i(t)$, differentiating with respect to e and setting $e = 0$ gives (3:16). Letting $Z_b^a = \partial Z^a/\partial t_b$, we find with the help of (3:3) and (3:6) that

$$\begin{aligned} \omega_i(\phi) &= (f_{ik} p_a^k + f_{ki}^c p_{ac}^k) Z^a + (f_{ki}^c p_a^k) Z_c^a \\ &= (\partial f_i/\partial t_a) Z^a + (\delta_a^c f_i) Z_c^a = \partial(f_i Z^a)/\partial t_a, \\ \omega_i^b(\phi) &= (f_{ik}^b p_a^k + f_{ik}^{bc} p_{ac}^k) Z^a + (f_{ik}^{bc} p_a^k) Z_c^a \\ &= (\partial f_i^b/\partial t_a) Z^a + (\delta_a^c f_i^b - \delta_a^b f_i^c) Z_c^a \\ &= \partial(f_i^b Z^a)/\partial t_a - f_i^c Z_c^b. \end{aligned}$$

Then a little manipulation shows that

$$\partial \omega_i^b(\phi)/\partial t_b = \partial(Z^a \partial f_i^b/\partial t_b)/\partial t_a.$$

Combining this equation with the last equation for $\omega_i(\phi)$ gives (3:17).

4. Notations to be employed in subsequent sections

For the sake of convenience of reference we list here the principal notations which we shall use later. We shall use the letters a, b, c, d, e, r and s , with or without subscripts as indices having the range 1 to m , while the letters i and k will be indices with the range 1 to n , and the letters j and l will be indices with the range 1 to $n - m$. Ordered collections of the letters a, b, c, d, e, i, k, r and s will be denoted by the corresponding Greek letter. Thus

$$\begin{aligned} \alpha &= (a_1, \dots, a_g), & \beta &= (b_1, \dots, b_g), & \gamma &= (c_1, \dots, c_h), \\ \delta &= (d_1, \dots, d_h), & \epsilon &= (e_1, \dots, e_h), & \iota &= (i_1, \dots, i_g), \\ \kappa &= (k_1, \dots, k_h), & \rho &= (r_1, \dots, r_g), & \sigma &= (s_1, \dots, s_h). \end{aligned}$$

Unless otherwise specified, the number of components in such a collection will always be g or h as given above.

If α and ι are given as above, then we shall use the symbol p_α^i to stand for the product $p_{a_1}^{i_1} p_{a_2}^{i_2} \dots p_{a_g}^{i_g}$, and the symbol f_α^i to stand for the repeated partial derivative of f with respect to $p_{a_1}^{i_1}, \dots, p_{a_g}^{i_g}$. By a summation such as $p_\alpha^i f_i^\beta$ we mean a summation over $i_1, \dots, i_g$ as each of these indices ranges independently from 1 to n .

For any m -rowed square matrix $B = (B_b^a)$ we shall use the symbol B_β^α to stand for the product $B_{b_1}^{a_1} \dots B_{b_g}^{a_g}$. In particular we shall use this notation for the identity matrix, so that $\delta_\beta^\alpha = 1$ in case $\alpha = \beta$ and $\delta_\beta^\alpha = 0$ otherwise.

Let α and β be as above and define D_β^α to be the g -rowed minor of the identity matrix whose rows are those given in the superscript α and whose columns are those given in the subscript β . Then it is clear that $D_\beta^\alpha = 1$ in case the $b_1, \dots, b_g$ are all distinct and the components of α form an even permutation of those of β , $D_\beta^\alpha = -1$ in case the b 's are distinct and the a 's are an odd permutation of the b 's, while $D_\beta^\alpha = 0$ otherwise. This quantity D_β^α is the generalized Kronecker symbol discussed in detail by Murnaghan [XIV, XV]. It may also be defined as the result obtained when the determinant $|B| = |B_b^a|$ is differentiated with respect to $B_{b_1}^{a_1}, \dots, B_{b_g}^{a_g}$ in turn and then B_b^a is set equal to δ_b^a . It is clear that if αQ denotes the collection resulting when a permutation Q is applied to the collection α and βR is a similar permutation of β , we have

$$(4:1) \quad D_{\beta R}^{\alpha Q} = (-1)^Q (-1)^R D_\beta^\alpha,$$

$$(4:2) \quad D_\beta^\alpha = \sum (-1)^Q \delta_\beta^{\alpha Q} = \sum (-1)^Q \delta_{\beta Q}^\alpha,$$

where $(-1)^Q = 1$ or -1 according as Q is an even or an odd permutation and the sums are taken for all permutations Q on g letters. We remark with Murnaghan that the following identities hold, in which $B = (B_b^a)$ is now an h -rowed square matrix:

$$(4:3) \quad D_\delta^\gamma = D_\gamma^\delta = D_\epsilon^\gamma \delta_\delta^\epsilon = (D_\epsilon^\gamma D_\delta^\epsilon)/h!,$$

$$(4:4) \quad |B| D_\delta^\gamma = D_\delta^\epsilon B_\epsilon^\gamma = D_\epsilon^\gamma B_\delta^\epsilon.$$

In the next section we shall have occasion to use a set of quantities $M_{\alpha;\gamma}^{\beta;\epsilon}$. These constants will be defined explicitly in Section 15. For the present it is sufficient to define them inductively as follows:

$$(4:5) \quad M_{a;c}^{b;\epsilon} = D_{ac}^{be},$$

$$(4:6) \quad M_{a;\gamma c}^{b;\epsilon \epsilon} = M_{a;\gamma}^{b;\epsilon} \delta_c^\epsilon - \delta_{\gamma ca}^{\epsilon b \epsilon},$$

$$(4:7) \quad M_{\alpha a;\gamma}^{\beta b;\epsilon} = M_{\alpha;d\delta}^{\beta;b\epsilon} M_{a;\gamma}^{d;\delta}.$$

5. Further consequences of the homogeneity condition

In this section we shall investigate the implications of the homogeneity condition (2:2) for the higher derivatives of f . In particular formulas generalizing (3:1), (3:5) and (3:6) will be obtained. The formula generalizing (3:1) is that stated in the following

THEOREM 5:1. *If $f(y, p)$ satisfies (2:2), then*

$$(5:1) \quad p_a^i f_i^\beta = D_\beta^\alpha f.$$

This follows at once upon differentiating (2:2) with respect to $B_{b_1}^{a_1}, \dots, B_{b_g}^{a_g}$ and setting $B_b^a = \delta_b^a$.

The formula generalizing (3:5) is stated in

THEOREM 5:2. *If $f(y, p)$ satisfies (2:2), then for each m -rowed square matrix B with positive determinant we have*

$$(5:2) \quad |B|^{g-1} f_i^\beta(y, pB) = C_a^\beta f_i^\alpha(y, p),$$

where C_a^b is the cofactor of B_a^b in the matrix B . Hence we also have, when $g = m$,

$$(5:3) \quad D_\beta^\alpha f_i^\beta(y, pB) = D_\beta^\alpha f_i^\beta(y, p).$$

If $g = 1$, we see from (3:5) that (5:2) is true. Assume that (5:2) is true for a fixed g and differentiate it with respect to p_a^i . We get

$$|B|^{g-1} B_c^a f_{ii}^{\beta\alpha}(y, pB) = C_a^\beta f_{ii}^{\alpha\alpha}(y, p).$$

If we multiply by C_a^b and sum for a , we find that

$$|B|^g f_{ii}^{\beta b}(y, pB) = C_{\alpha\alpha}^{\beta b} f_{ii}^{\alpha\alpha}(y, p).$$

This completes the induction and proves (5:2). Multiplication of (5:2) for $g = m$ by D_β^γ with $\gamma = (c_1, \dots, c_m)$ yields, when we apply (4:4) to the matrix C ,

$$|B|^{m-1} D_\beta^\gamma f_i^\beta(y, pB) = D_a^\gamma |C| f_i^\alpha(y, p),$$

whence (5:3) follows, since $|C| = |B|^{m-1} > 0$.

The generalization of (3:6) is stated in

THEOREM 5:3. *If $f(y, p)$ satisfies (2:2), then*

$$(5:4) \quad p_a^i f_{ik}^{\beta\epsilon} = M_{a;\gamma}^{\beta;\epsilon} f_k^\gamma,$$

where $M_{a;\gamma}^{\beta;\epsilon}$ is defined by equations (4:5), (4:6), (4:7).

For $g = h = 1$ the equation (5:4) becomes, when we use (4:5), precisely (3:6) and is therefore true. Assume that (5:4) is true for $g = 1$ and a fixed h . Differentiating with respect to p_e^k gives

$$p_a^i f_{ikk}^{b\epsilon\epsilon} + \delta_a^e f_{kk}^{b\epsilon} = M_{a;\gamma}^{b;\epsilon} f_{kk}^{\gamma\epsilon},$$

$$p_a^i f_{ikk}^{b\epsilon\epsilon} = (M_{a;\gamma}^{b;\epsilon} \delta_c^e - \delta_{\gamma ca}^{\epsilon b\epsilon}) f_{kk}^{\gamma c} = M_{a;\gamma c}^{b;\epsilon\epsilon} f_{kk}^{\gamma c},$$

by virtue of (4:6). Hence (5:4) is true for $g = 1$ and all h . Assume it true for a fixed g and all h . Then we have

$$p_{\alpha\alpha}^{ii} f_{iik}^{\beta b\epsilon} = M_{\alpha;\delta\delta}^{\beta;b\epsilon} (p_a^i f_{iik}^{d\delta}) = M_{\alpha;\delta\delta}^{\beta;b\epsilon} M_{a;\gamma}^{d;\delta} f_k^\gamma = M_{\alpha a;\gamma}^{\beta b;\epsilon} f_k^\gamma,$$

when we use (5:4) for $g = 1$ and (4:7). This completes the induction and proves (5:4) for all values of g and h .

6. The case when $n = m + 1$

In this case certain additional consequences of the homogeneity condition are obtainable, and we shall find these results useful in comparing the work of various authors. By Theorem 2:2 we have $f(y, p) = G(y, A)$, where G satisfies

$$(6:1) \quad G(y, KA) = K G(y, A)$$

for every positive number K . Then it follows easily that we have

$$(6:2) \quad A_i G_i = G \quad (G_i = \partial G / \partial A_i),$$

$$(6:3) \quad A_i G_{ik} = 0 \quad (G_{ik} = \partial G_i / \partial A_k = G_{ki}).$$

We remark with Radon [XVIII] that (6:3) implies the existence of quantities $\theta^{ab} = \theta^{ba}$ such that

$$(6:4) \quad G_{ik} = p_a^i \theta^{ab} p_b^k.$$

In order to derive a formula for the derivatives G_i in terms of the derivatives f_i^a we remark that the definition of A_i as the cofactor of x_i in the matrix $(p_a^i x_i)$ enables us to write (6:2) as

$$(6:5) \quad f(y, p) = G(y, A) = |p_a^i G_i|.$$

Then $f_i^a = G_k \partial A_k / \partial p_a^i$ is the cofactor of p_a^i in the above determinant. It follows that $f_i^a G_i = 0$. We also have the equations $p_a^i A_i = 0$, expressing the perpendicularity of the normal vector to the tangent vectors. Using (3:1) and the relations derived above we readily find that $|p_a^i G_i| |f_i^a A_i| = f^{m+1}$. If $f \neq 0$, we therefore have $A_i F_i = f^m$, where F_i is the cofactor of A_i in the determinant $|f_i^a A_i|$. With this definition of F_i we see that $f_i^a F_i = 0$. Consequently we have

$$A_i (F_i - G_i f^{m-1}) = 0, \quad f_i^a (F_i - G_i f^{m-1}) = 0.$$

Since $|f_i^a A_i| = f^m \neq 0$, we conclude that

$$(6:6) \quad F_i = G_i f^{m-1},$$

at least at all points where $f \neq 0$. If $m = 1$, the relation (6:6) is trivially established in the form $F_i = G_i$, regardless of the vanishing of f . If $m > 1$ and $f = 0$, then equations (3:1) show that the matrix (f_i^a) has rank at most 1, so that $F_i = 0$. In this case both sides of (6:6) are zero, so that the equation (6:6) holds without restriction. Thus we have proved the first part of

THEOREM 6:1. *If $n = m + 1$, so that $f(y, p) = G(y, A)$, then the first derivatives of G with respect to the variables A_i satisfy (6:6) when the F_i are the cofactors of A_i in the matrix $(f_i^a A_i)$. If we define a symbol f_β^i by the equation*

$$(6:7) \quad f_\beta^i = f_{i_1}^{b_1} f_{i_2}^{b_2} \cdots f_{i_o}^{b_o},$$

we have, at each point where $f \neq 0$,

$$(6:8) \quad G_i = (-1)^{n+i} f^{1-m} D_{\alpha_0}^\alpha f_\alpha^{\kappa_i}, \quad (i \text{ not summed}),$$

where $\alpha_0 = (1, 2, \cdots, m)$, $\kappa_i = (1, \cdots, i-1, i+1, \cdots, m)$ and $\alpha = (\alpha_1, \cdots, \alpha_m)$. The symbol f_β^i is related to the symbol f_i^a by the identity

$$(6:9) \quad f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon = h! D_\epsilon^\gamma f_\kappa^\epsilon,$$

where γ , ϵ and κ are defined in Section 4. Therefore we also have

$$(6:10) \quad G_i = (-1)^{n+i} (D_{\alpha_0}^\alpha f_\alpha^{\kappa_i}) / m! \quad (i \text{ not summed}).$$

We get (6:8) by expanding the cofactor F_i according to the rule given in (4:4). We observe that (6:9) is trivially true for $h = 1$. By (5:4) and (3:1) we have

$$(6:11) \quad p_d^{k_1} f_\kappa^\epsilon = M_{d;\delta}^{\epsilon_1;\epsilon'} f_\kappa^\delta, \quad p_d^{k_1} f_\epsilon^\kappa = \delta_d^{\epsilon_1} f f_\epsilon^{\kappa'},$$

where the sum is for the k_1 occurring explicitly in $p_d^{k_1}$ and implicitly in the subscript κ . Here $\kappa' = (k_2, \dots, k_h)$, $\epsilon' = (e_2, \dots, e_h)$ and $\delta = (d_1, \dots, d_{h-1})$. It will be shown in Section 15 that

$$(6:12) \quad D_{\rho\sigma}^{\beta\epsilon} M_{\alpha;\gamma}^{\rho;\sigma} = \frac{(g+h)!}{h!} D_{\alpha\gamma}^{\beta\epsilon}.$$

Using this formula and equation (6:11) we find that

$$p_d^{k_1} (f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon - h! D_\epsilon^\gamma f_\kappa^\kappa) = h f (f^{h-2} D_{d\delta}^\gamma f_\kappa^\delta - (h-1)! D_{d\delta}^\gamma f_\delta^{\kappa'}).$$

Assume now that (6:9) is true for $h-1$. Since we have from the definition of a determinant that

$$D_{d\delta}^\gamma = \sum_{\lambda=1}^h (-1)^{\lambda-1} \delta_d^{\epsilon_\lambda} D_\delta^{\gamma_\lambda},$$

where $\gamma_\lambda = (c_1, \dots, c_{\lambda-1}, c_{\lambda+1}, \dots, c_h)$, and since by the hypothesis of the induction we have

$$f^{h-2} D_\delta^{\gamma_\lambda} f_\kappa^\delta - (h-1)! D_\delta^{\gamma_\lambda} f_\delta^{\kappa'} = 0,$$

we conclude that

$$p_d^{k_1} (f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon - h! D_\epsilon^\gamma f_\kappa^\kappa) = 0.$$

Since the quantity in the parenthesis is changed in sign by an interchange of two of the components of κ , we conclude that there exist quantities C^γ such that

$$f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon - h! D_\epsilon^\gamma f_\kappa^\kappa = C^\gamma A_{k_1} A_{k_2} \cdots A_{k_h}.$$

The right hand side of this equation is unaltered by an interchange of two of the components of κ while the left hand side is changed in sign. Hence both sides are zero. This completes the induction and proves (6:9). Then (6:9) for $h = m$ and (6:8) imply (6:10).

That the formula (6:9) does not hold in general if $n \neq m+1$ (or $m \neq 1$) is shown by the following example with $m = 2$, $n = 4$. Let

$$f(y, p) = [(p_1^1 p_2^2 - p_1^2 p_2^1)(p_1^3 p_2^4 - p_1^4 p_2^3)]^{\frac{1}{2}}.$$

Then $f(y, p)$ satisfies (2:2) and by actual computation we find that

$$f(f_{12}^{12} - f_{12}^{21}) - 2(f_1^1 f_2^2 - f_1^2 f_2^1) = \frac{1}{4}(p_1^3 p_2^4 - p_1^4 p_2^3) \neq 0.$$

This example also shows that the functions C_{ij}^{ab} of Theorem 3:2 are not identically zero in f , although this is true for $n = m+1$ and for $m = 1$.

It is clear that a matrix $\bar{q}$ satisfying (2:3) will have one column and that we may take $q_i^i = A_i$. By (3:10) we have

$$(6:13) \quad f_{ik}^{ab} + f_{ik}^{ba} = 2R_{11}^{ab}A_iA_k.$$

THEOREM 6:2. *The second derivatives of G with respect to the variables A_i are given by (6:4) with $\theta^{ab} = R_{11}^{ab}$.*

Since $n = m + 1$, we find that

$$\begin{aligned} p_c^\mu \partial A_\mu / \partial p_a^i &= -\delta_a^c A_i & (\mu, \nu &= 1, \dots, n), \\ \partial^2 A_\mu / \partial p_a^i \partial p_b^k + \partial^2 A_\mu / \partial p_b^i \partial p_a^k &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} f_{ik}^{ab} + f_{ik}^{ba} &= G_{\mu\nu}[(\partial A_\mu / \partial p_a^i)(\partial A_\nu / \partial p_b^k) + (\partial A_\mu / \partial p_b^i)(\partial A_\nu / \partial p_a^k)] \\ &= \theta^{cd} p_c^\mu p_d^\nu [(\partial A_\mu / \partial p_a^i)(\partial A_\nu / \partial p_b^k) + (\partial A_\mu / \partial p_b^i)(\partial A_\nu / \partial p_a^k)] \\ &= 2\theta^{ab} A_i A_k. \end{aligned}$$

Comparison of this last equation with (6:13) completes the proof of the theorem.

7. Analytic formulation of the problem

Because of the results established in Section 2, we may state our parametric multiple integral problem in the calculus of variations in the following manner. Let $\mathfrak{R}$ be a region in $(n + nm)$ -dimensional space of points (y, p) with the property that if (y, p) is in $\mathfrak{R}$, then p has rank m and the point (y, pB) is also in $\mathfrak{R}$ for every m -rowed square matrix B with positive determinant. Let $f(y, p)$ be a function of class C^m on $\mathfrak{R}$ which satisfies the homogeneity condition (2:2).

By an admissible region T , we mean one in m -dimensional t -space whose boundary B can be subdivided into a finite number of non-singular $(m - 1)$ -dimensional subsets of class C' whose intersections form a set E with the property that there exist positive numbers q and s with $s < m - 1$ such that, given a number $r > 0$, the quantity q/r^s is an upper bound of the number of points in a set E^* contained in E and chosen so that the distance between any two points of E^* is at least r . Such regions were considered by Carson [VI] and he showed that Green's lemma is valid for them.

By an admissible set of functions $y^i(t)$, we mean a set of functions of class C' on $T + B$, where T is an admissible region, for which the point $[y(t), p(t)]$ is in $\mathfrak{R}$ for t in $T + B$. We will say that two admissible sets of functions, $y^i(t)$ on $T + B$ and $Y^i(u)$ on $U + C$, are equivalent in case there exists a 1-1 transformation $t_a = t_a(u)$ of class C' between $T + B$ and $U + C$ for which $|\partial t_a / \partial u_b|$ is positive and such that $y^i(t(u)) = Y^i(u)$ on $U + C$. This relation is clearly reflexive, symmetric and transitive, and may therefore be used to classify admissible sets of functions into maximal classes of equivalent sets. We now define an admissible variety to be such a class of equivalent admissible sets of functions. The results of Section 2 show that the integral (1:1) defines a number which is the

same for all elements of a class of equivalent admissible sets of functions. Hence it depends only on the variety and we may set

$$(7:1) \quad I(V) = \int_T f(y, p) dT.$$

The calculus of variations problem to be considered may now be stated as the problem of finding that admissible variety V which gives the integral (7:1) its minimum value in the class of admissible varieties having the same boundary.

8. Necessary conditions associated with the first variation

Let V be an admissible variety represented by a set of functions $y^i(t)$ for t in $T + B$ and let L be its boundary. We say that the functions $\eta^i(t)$ define an admissible variation in case they are of class C' on $T + B$. The first variation of $I(V)$ along V then has the form

$$(8:1) \quad I_1(\eta) \equiv \int_T (f_i \eta^i + f_i^a \pi_a^i) dT \equiv \int_T \Phi(\eta) dT,$$

where $\pi_a^i = \partial \eta^i / \partial t_a$. If V is a minimizing variety, it is clear that $I_1(\eta) = 0$ for all admissible variations $\eta^i(t)$ such that $\eta^i(t) \equiv 0$ on B . Carson [VI] has shown that this will occur if and only if the Haar-Coral equations

$$(8:2) \quad \int_{T_0} f_i dT = \int_{B_0} f_i^a m_a dB$$

hold for every cube T_0 which together with its boundary B_0 is contained in T , where the m_a are the direction cosines of the outer normal to B_0 . If moreover the functions $y^i(t)$ are of class C'' , then the equations (8:2) hold if and only if the Euler-Lagrange equations

$$(8:3) \quad T_i \equiv f_i - \partial f_i^a / \partial t_a = 0$$

are satisfied at each point of T . Solutions of these equations are called extremals. The equations (8:3) are not all independent, since (3:15) holds. But we see from (3:15) that there exist functions S_j ($j = 1, \dots, n - m$) such that $T_i = S_j q_j^i$, so that a variety is an extremal if and only if $S_j = 0$.

Suppose now that V is of class C'' . Then there exist functions $u_j^i(t)$ of class C' on T such that the matrix (p, u) is non-singular. In fact, we could take $u = q$. For any admissible variations $\eta^i(t)$ the equations

$$(8:4) \quad \eta^i = p_a^i Z^a + u_j^i w^j$$

can be solved to obtain unique functions $Z^a(t)$ and $w^j(t)$ of class C' on T . Moreover, $\eta^i(t) \equiv 0$ on B if and only if $Z^a(t) \equiv w^j(t) \equiv 0$ on B . By straightforward calculation we find that $\Phi(\eta) = \Phi(uw) + \partial(Z^a f) / \partial t_a$. Then if $\eta^i(t) \equiv 0$ on B we see from Green's lemma that

$$(8:5) \quad I_1(\eta) = \int_T (M_j w^j + N_b^i w_b^i) dT \quad (w_b^i = \partial w^i / \partial t_b),$$

where M_j and N_j^b are defined by

$$(8:6) \quad M_j = f_i u_j^i + f_i^b u_{jb}^i, \quad N_j^b = f_i^b u_j^i \quad (u_{jb}^i = \partial u_j^i / \partial t_b).$$

THEOREM 8:1. *Along an admissible variety V of class C'' the first variation is expressible in the form (8:5), where $w = [(q'u)^{-1}q]\eta$. The vanishing of the integral (8:5) for all functions $w^i(t)$ of class C' such that $w^i(t) \equiv 0$ on B is equivalent to the vanishing of the integral (8:1) for all admissible variations $\eta^i(t)$ such that $\eta^i(t) \equiv 0$ on B .*

We need merely verify the formula for w . The equation (8:4) can be written in matrix form as $pZ + uw = \eta$, whence multiplication by q' , remembering that $q'p = 0$, shows that $q'uw = q'\eta$. The matrix $q'u$ is non-singular since both q and u have rank $n - m$, and consequently our formula for w is true.

The Euler-Lagrange equations for the integral (8:5) are

$$f_i u_j^i + f_i^b u_{jb}^i - \partial(f_i^a u_j^i) / \partial t_a = 0,$$

and these reduce immediately to $T_i u_j^i = S_i(q_i^i u_j^i) = 0$, equations which are equivalent to $S_j = 0$ since the matrix $q'u$ has rank $n - m$.

Ridge conditions could be stated for our problem. Since these conditions are precisely the same as for the non-parametric multiple integral problem treated by Powell [XVI] or Carson [VI], we shall not formulate them precisely here.

9. The conditions of Weierstrass and Legendre

The direct generalization of the necessary condition of Weierstrass for simpler problems would be to assert that the function

$$(9:1) \quad E_0(y, p, P) = f(y, P) - f(y, p) - (P_a^i - p_a^i) f_i^a(y, p)$$

is non-negative for every matrix P and for all points (y, p) belonging to the minimizing variety V . But a simple example (see Bliss [I, p. 80]) can be given to show that this condition is in general false, although McShane [XII], Weyl [XIX] and Graves [VII] have each proved that E_0 is non-negative whenever the matrix $P - p$ has rank one. Caratheodory [V] and Boerner [III, IV] have considered an E -function having the form

$$(9:2) \quad E_c(y, p, P) = f(y, P) - f^{1-m} | \delta_b^a f + (P_a^i - p_a^i) f_i^b |,$$

assuming for their theory that $f > 0$. By expanding the determinant as a polynomial in f we see that $E_c = E_0$ whenever $P - p$ has rank one. They found their function useful in making sufficiency proofs, but they did not consider the necessity of $E_c \geq 0$. We shall give later on an example similar to that already quoted and show that $E_c \geq 0$ is not necessary for a minimum. For our parametric problem E_c reduces to the expression

$$(9:3) \quad E_c(y, p, P) = f(y, P) - f^{1-m} | P_a^i f_i^b |,$$

since (3:1) holds. Computing the determinant we find that

$$(9:4) \quad E_c(y, p, P) = f(y, P) - f^{1-m} D_\gamma^\beta P_\beta^\iota f_\gamma^\iota / m!,$$

in which ι, β and γ are m -partite and f_γ^ι is the quantity defined in (6:7).

For the purpose of making sufficiency proofs we must have $E(y, p, P) = f(y, P) - H(y, p, P)$, where $H(y, P(y), p)$ has the form of the integrand of an invariant integral. It will be shown in Section 13 that H is necessarily of the form $H(y, p, P) = P_\beta^\iota C_\iota^\beta(y, p)$, where ι and β are m -partite and C_ι^β is skew in ι and β , i.e., an interchange of two of the components of either ι or β changes the sign of C_ι^β . Hence we must have

$$(9:5) \quad E(y, p, P) = f(y, P) - P_\beta^\iota C_\iota^\beta(y, p).$$

It then follows that we have

$$(9:6) \quad E(y, p, PB) = |B| E(y, p, P)$$

for every m -rowed square matrix B with a positive determinant. But we also require that E satisfy

$$(9:7) \quad E(y, pB, P) = E(y, p, P), \quad E(y, p, p) = 0.$$

It is clear that the first equation in (9:7) is equivalent to

$$(9:8) \quad C_\iota^\beta(y, pB) = C_\iota^\beta(y, p).$$

Moreover, by virtue of (9:6), the second equation in (9:7) holds if and only if $E(y, p, pB) = 0$. Differentiation of this relation with respect to $B_{a_1}^{b_1}, \dots, B_{a_m}^{b_m}$ gives, when we use (5:1),

$$p_\beta^\iota f_\iota^\alpha = m! p_\beta^\iota C_\iota^\alpha.$$

Defining a new quantity F_ι^γ by the equation

$$(9:9) \quad F_\iota^\gamma = D_\alpha^\gamma f_\iota^\alpha / m!,$$

we see from (5:1) that $p_\beta^\iota f_\iota^\gamma = p_\beta^\iota F_\iota^\gamma$. Hence the second equation in (9:7) holds if and only if

$$(9:10) \quad p_\beta^\iota F_\iota^\gamma = m! p_\beta^\iota C_\iota^\gamma.$$

The equations (9:8) and (9:10) do not suffice to determine the form of the E -function uniquely. For it is obvious that $C_\iota^\gamma = F_\iota^\gamma / m!$ satisfies (9:10) and it also satisfies (9:8) by virtue of (5:3). This choice of C_ι^γ gives an E -function of the form

$$(9:11) \quad E_1(y, p, P) = f(y, P) - P_\beta^\iota D_\gamma^\beta f_\iota^\gamma / (m!)^2.$$

But the function $E_c(y, p, P)$ given by (9:4) also has functions $C_\iota^\beta = f^{1-m} D_\gamma^\beta f_\gamma^\iota / m!$ which are easily seen to satisfy (9:8) and (9:10). For $n = m + 1$ we see from the definitions (9:4) and (9:11) and the identity (6:9) that $E_c = E_1$. The remarks following the proof of Theorem 6:1 show that E_c is not in general equal to E_1 .

For the case $n = m + 1$, Radon [XVIII] has defined an E -function in terms of the function $G(y, A)$. This function is

$$(9:12) \quad E_r(y, A, A^*) = G(y, A^*) - A_h^* G_h(y, A) \quad (h = 1, \dots, n),$$

and it should be remarked that McShane [XIII] has proved that $E_r \geq 0$ is necessary for a minimum. If A_i^* are the minors corresponding to the matrix P , then we have, using Theorem 6:1,

$$A_h^* = (-1)^{n-h} D_{\alpha_0}^{\alpha} P_{\alpha}^{\kappa_h}, \quad G_h = (-1)^{n+h} D_{\beta}^{\alpha} f_{\kappa_h}^{\beta} / m!,$$

where $\alpha_0 = (1, 2, \dots, m)$, $\kappa_h = (1, \dots, h-1, h+1, \dots, n)$, and α and β are m -partite. Thus we see that

$$E_r = f(y, P) - P_{\alpha}^{\kappa_h} F_{\kappa_h}^{\alpha} = f(y, P) - P_{\alpha}^{\iota} F_{\iota}^{\alpha} / m!.$$

Consequently, for $n = m + 1$ we have $E_1 = E_c = E_r$.

The function E_1 can be exhibited in a form which more clearly displays some of its properties. This form is

$$(9:13) \quad E_1(y, p, P) = f(y, P) - f(y, p) - \sum_{g=1}^m \frac{1}{g!} (P - p)_{\alpha}^{\iota} F_{\iota}^{\alpha},$$

where ι and α are now g -partite as in Section 4, $(P - p)_{\alpha}^{\iota}$ is the product of the quantities $(P_{\alpha_{\lambda}}^{\iota_{\lambda}} - p_{\alpha_{\lambda}}^{\iota_{\lambda}})$ ($\lambda = 1, \dots, g$), and $F_{\iota}^{\alpha} = D_{\beta}^{\alpha} f_{\iota}^{\beta} / g!$. To prove this formula, one expands the function $H(y, p, P)$ about the point (y, p, p) by Taylor's theorem and simplifies the result by employing (5:4) and (6:12).

We have already seen that $E_c = E_0$ if $P - p$ has rank one. But in this case we see that for $g > 1$ the product $(P - p)_{\alpha}^{\iota}$ is unaltered by any permutation of ι . Hence $(P - p)_{\alpha}^{\iota} F_{\iota}^{\alpha} = 0$ if $g > 1$. For $g = 1$ we have $F_{\iota}^{\alpha} = f_{\iota}^{\alpha}$, and thus we conclude from (9:13) that $E_1 = E_0 = E_c$ in case $P - p$ has rank one.

The function E_1 is not open to the objection raised against E_0 that it does not vanish identically for integrands of invariant integrals. In fact, if $f(y, p) = p_{\beta}^{\iota} C_{\iota}^{\beta}$ where ι and β are m -partite and C_{ι}^{β} is skew in ι and β , then $f_{\iota}^{\gamma} = m! C_{\iota}^{\gamma}$, $D_{\gamma}^{\beta} f_{\iota}^{\gamma} = (m!)^2 C_{\iota}^{\beta}$, so that we conclude from (9:11) that $E_1(y, p, P) = 0$.

This result is not true for the function E_c . To see this take $m = 2$, $n = 4$ and let f be given by

$$f = p_1^1 p_2^2 - p_1^2 p_2^1 + p_1^3 p_2^4 - p_1^4 p_2^3.$$

Then f satisfies the homogeneity condition (2:2) and is actually the integrand of an invariant integral. We find by straightforward calculation that

$$E_c = \sum_{i < k} [(P_1^i - p_1^i) (P_2^k - p_2^k) - (P_2^i - p_2^i) (P_1^k - p_1^k)] A_{ik},$$

in which $A_{ik} = p_1^h p_2^j - p_2^h p_1^j$, where h and j are selected so that (i, k, h, j) is an even permutation of $(1, 2, 3, 4)$. Therefore we see that $E_c \neq 0$. It follows that $E_c \geq 0$ is not necessary for a minimum.

Similar results may be stated for the Legendre condition. The quadratic form

$$Q(y, p, v) = [f_{ik}^{ab}(y, p) - H_{ik}^{ab}(y, p, p)] v_a^i v_b^k,$$

where $H_{ik}^{ab} = \partial^2 H / \partial P_a^i \partial P_b^k$, is the Legendre quadratic form belonging to the E -function $E(y, p, P) = f(y, P) - H(y, p, P)$. As a consequence of $E_0 \geq 0$ for all P such that $P - p$ has rank one, we conclude that

$$(9:14) \quad Q_0(y, p, v) = f_{ik}^{ab}(y, p) v_a^i v_b^k \geq 0$$

for all matrices v of rank one. This result has also been proved independently of the Weierstrass condition by Boerner [IV].

Using the identities of Theorem 3:2, we can prove with the help of a little computation which will be omitted that the Legendre quadratic forms for the E -functions E_1 and E_c are respectively

$$\begin{aligned} Q_1(y, p, v) &= R_{il}^{ab}(q_i^i v_a^i)(q_i^k v_b^k), \\ Q_c(y, p, v) &= f^{-1}(f R_{il}^{ab} + C_{il}^{ab})(q_i^i v_a^i)(q_i^k v_b^k). \end{aligned}$$

Therefore Q_1 and Q_c vanish identically when $v = pB$, although we see from (3:8) that this property is not possessed by Q_0 . If $f = p_\beta^i C_i^\beta(y)$, where ι and β are m -partite and C_i^β is skew in ι and β , then Q_1 vanishes identically but this need not be true for either Q_0 or Q_c .

10. The second variation

Let V be a fixed extremal variety. The second variation of $I(V)$ along V is the expression

$$(10:1) \quad I_2(\eta) = \int_T 2\omega(t, \eta, \pi) dT,$$

where the function 2ω was defined in equation (3:13). If V is a minimizing variety we must have $I_2(\eta) \geq 0$ in the class of admissible variations $\eta^i(t)$ which vanish on B . The Haar-Coral accessory equations are

$$(10:2) \quad \int_{T_0} \omega_i dT = \int_{B_0} \omega_i^a m_a dB.$$

The Euler equations for $I_2(\eta)$ are called the Jacobi equations and are $J_i(\eta) = 0$, where $J_i(\eta)$ was defined in (3:14). Since V is an extremal we see from (3:16) that $p_a^i J_i(\eta) = 0$. Thus there exist functions $J^j(\eta)$ such that $J_i(\eta) = J^j(\eta) q_i^j$ so that $J_i(\eta) = 0$ if and only if $J^j(\eta) = 0$.

Assuming now that V is of class C''' we shall subject the second variation to the same transformation used in Section 8 on the first variation. We set

$$\eta^i = p_a^i Z^a + u_j^i w^j = \phi^i + u_j^i w^j, \quad \phi^i = p_a^i Z^a.$$

Since equation (3:17) implies that $J_i(\phi) = 0$, we find after a little calculation that

$$2\omega(\eta) = 2\omega(uw) + \partial[(2u_j^i w^j + \phi^i)\omega_i^a(\phi)]/\partial t_a.$$

If $\eta^i = 0$ on B , then $w^j = \phi^i = 0$ on B , and by Green's lemma we have

$$I_2(\eta) = \int_T 2\omega(uw) dT.$$

If we set $u_j = (u_j^1, \dots, u_j^n)$, some more computation shows that

$$2\omega(uw) = w^j w^l F_{jl} + w_b^j w_a^l f_{ki}^{ab} u_j^i u_l^k + \partial[w^j w^l u_i^k \omega_k(u_j)]/\partial t_a,$$

where $w_b^j = \partial w^j/\partial t_b$ and F_{jl} is defined by the equation

$$(10:3) \quad F_{jl} = u_l^k J_k(u_j).$$

Another application of Green's lemma suffices to prove the following

THEOREM 10:1. *The second variation (10:1) is expressible in the form*

$$(10:4) \quad I_2(\eta) = \int_T (w^j w^l F_{jl} + w_b^j w_a^l f_{ki}^{ab} u_j^i u_l^k) dT,$$

in which the functions F_{jl} are defined by (10:3) and $w = [(q'u)^{-1}q]\eta$. The non-negativeness of (10:4) for arbitrary functions $w^j(t)$ of class C' which vanish on B is equivalent to the non-negativeness of (10:1) for arbitrary admissible variations $\eta^i(t)$ vanishing on B . The Jacobi equations are

$$(10:5) \quad \partial(w_b^j f_{ki}^{ab} u_j^i u_l^k)/\partial t_a - \frac{1}{2} w^j (F_{jl} + F_{li}) = 0.$$

Both this theorem and Theorem 8:1 are generalizations of theorems stated by Bliss [II] for the case $m = 2$, $n = 3$. When $n = m + 1$, one obtains a further simplification as is seen in the

COROLLARY. *If $n = m + 1$ and we take $u_i^i = A_i/H$, where $H^2 = A_i A_i$, then the second variation may be written in the form*

$$(10:6) \quad I_2(\eta) = \int_T (w^2 F_1 + w_a w_b R^{ab}) dT \quad (w = A_i \eta^i/H^2),$$

where $R^{ab} = R_{11}^{ab}H$, the quantity R_{11}^{ab} being defined by (6:13), and where $F_1 = J^1(A/H)H$, the function $J^1(\eta)$ being determined by the equations $J_i(\eta) = J^1(\eta)A_i$. The Jacobi equation is

$$(10:7) \quad \partial(w_a R^{ab})/\partial t_b = w F_1.$$

11. The Jacobi condition

Let us call an admissible variation $\eta^i(t)$ normal in case it satisfies in T the equations

$$(11:1) \quad p_a^i \eta^i(t) = 0,$$

where the p_a^i are the derivatives belonging to the minimizing variety V , assumed now to be of class C' . By an admissible subregion T_0 of T we mean a subregion of T whose boundary B_0 satisfies the conditions imposed on B in Section 7, and we denote by E_0 the set of intersections of the pieces of B_0 . Using the methods

which Raab [XVII] employed to prove a Jacobi condition in the parametric problem for varieties of class C'' and which Carson [VI] used in the non-parametric problem for varieties of class C' , we can prove the following

THEOREM 11:1. *If V is a minimizing variety of class C' on which $Q_0(y, p, v) > 0$ for all matrices v of rank one such that $p'v = 0$, then there can exist no solution $\eta^i = z^i(t)$ of class C' of the equations*

$$(11:2) \quad \int_{T'} \omega_i dT = \int_{B'} \omega_i^a m_a dB$$

for all cubes T' contained in an admissible subregion T_0 , such that the variation $z^i(t)$ is normal, vanishes on B_0 , and has first partial derivatives not all zero at some point t_0 of $B_0 - E_0$ in T .

For if such a solution existed, we could show precisely as in Carson [VI] that the matrix $v = (\partial z^i(t_0)/\partial t_a)$ would have rank one and would also satisfy $Q_0(y, p, v) = 0$. By hypothesis we have $p_a^i z^i = 0$. Since $z^i(t_0) = 0$, it follows that $p_a^i(t_0)v_b^i = 0$. This contradiction proves the theorem.

12. The definition of a field

By a field we mean a region $\mathfrak{F}$ of y -space on which is defined a set of slope functions $P_a^i(y)$ of class C' . Moreover, we assume the existence of an integral

$$(12:1) \quad I^*(V) = \int_T H(y, p) dT$$

which is invariant in the sense that it has the same value for all admissible varieties lying in $\mathfrak{F}$ and having the same boundary. The function H is also required to satisfy

$$(12:2) \quad H(y, P(y)) = f(y, P(y)), \quad H_i^a(y, P(y)) = f_i^a(y, P(y)),$$

where $H_i^a = \partial H / \partial p_a^i$. Since every variety V is a minimizing variety for an invariant integral, and therefore satisfies the Euler-Lagrange equations for the integral, a familiar argument based on equations (12:2) suffices to show that every solution of the equations

$$(12:3) \quad \partial y^i / \partial t_a = P_a^i(y)$$

is an extremal for the integral (7:1). Such an extremal is called an extremal of the field. If V is an extremal of the field, then

$$(12:4) \quad I(V) = I^*(V).$$

13. Invariant integrals

In order to determine whether a region $\mathfrak{F}$ with given slope functions forms a field, it is necessary to determine whether or not the integral (12:1) is invariant. The following theorem is an analogue for parametric problems of the corresponding theorem for non-parametric problems proved by Landers [IX].

THEOREM 13:1. A parametric integral (12:1) is invariant only if the integrand function $H(y, p)$ has the form

$$(13:1) \quad H(y, p) = p_{\alpha}^{\iota} C_{\iota}^{\alpha}(y) \quad (\iota, \alpha \text{ } m\text{-partite}),$$

where $C_{\iota}^{\alpha}(y)$ is skew in ι and α and satisfies the partial differential equations

$$(13:2) \quad \partial C_{\iota}^{\alpha} / \partial y^k = \partial C_{\iota_b}^{\alpha} / \partial y^{i_b}$$

in which $\iota_b = (i_1, \dots, i_{b-1}, k, i_{b+1}, \dots, i_m)$. These conditions are also sufficient that $I^*(V) = I^*(V_0)$ hold for every pair of varieties V and V_0 with the same boundary and admitting representations with the same parameters, provided that the region $\mathfrak{F}$ is convex.

If we assume that $I^*(V)$ is invariant, the Euler equations

$$H_k = H_{ki}^b p_b^i + H_{ki}^{bc} p_{bc}^i$$

must be identities in y^i, p_a^i and $p_{ab}^i = p_{ba}^i$. Thus we have

$$(13:3) \quad H_k = H_{ki}^b p_b^i$$

and also $H_{ki}^{bc} + H_{ki}^{cb} = 0$. From this result it is seen by differentiation that if κ and β are g -partite, then $H_{\kappa}^{\beta}(y, p)$ is skew in κ and β . Therefore $H(y, p)$ must be a polynomial of degree at most m in p_a^i ,

$$H(y, p) = C(y) + \sum_{g=1}^m p_{\beta}^{\kappa} C_{\kappa}^{\beta}(y).$$

Then the equations $H_{\kappa}^{\beta}(y, 0) = g! C_{\kappa}^{\beta}(y)$ show that $C_{\kappa}^{\beta}(y)$ is skew in β and κ . Since $I^*(V)$ is a parametric integral, we must have $H(y, pB) = |B| H(y, p)$ for every m -rowed square matrix B with positive determinant. Take $B = 2I$, where I is the identity matrix. Then we have as an identity in y^i, p_a^i

$$C(y) + \sum_{g=1}^m 2^g p_{\beta}^{\kappa} C_{\kappa}^{\beta}(y) = 2^m C(y) + \sum_{g=1}^m 2^m p_{\beta}^{\kappa} C_{\kappa}^{\beta}(y).$$

Equating coefficients shows that $C(y) = 0, C_{\kappa}^{\beta}(y) = 0$ for $g < m$, and thus we have proved that H must have the form specified in (13:1). For this function the equation (13:3) takes the form

$$p_{\alpha}^{\iota} \partial C_{\iota}^{\alpha} / \partial y^k = m p_{b\sigma}^{i_b} \partial C_{k\rho}^{b\sigma} / \partial y^i,$$

where $\rho = (r_1, \dots, r_{m-1}), \sigma = (s_1, \dots, s_{m-1})$, the r 's and s 's being (for the purposes of this section only) indices ranging from 1 to n . Equating coefficients of p_{α}^{ι} on both sides of this equation shows that (13:2) must hold.

To prove the sufficiency, we first prove two lemmas.

LEMMA 13:1. If $\mathfrak{F}$ is convex and the functions $C_{\alpha}^{\iota}(y)$ satisfy the conditions of Theorem 13:1, then there exist functions $E^{\kappa}(y)$ with $\kappa = (k_1, \dots, k_{m-1})$ such that

$$(13:4) \quad (-1)^{c-1} \partial E^{\iota'_c} / \partial y^{i_c} = C_{\iota}^{\alpha_0}$$

holds, where $\alpha_0 = (1, \dots, m), \iota'_c = (i_1, \dots, i_{c-1}, i_{c+1}, \dots, i_m)$.

It can be assumed that the origin is in $\mathfrak{F}$. Let

$$V_{\iota}^l(y) = \sum C_{\iota}^{\alpha_0}(y^{\rho}),$$

where the sum is taken for all possible choices of the collection $\rho = (r_1, \dots, r_l)$ of distinct integers such that ρ and ι have no component in common, and $C_{\iota}^{\alpha_0}(y^{\rho})$ stands for the function obtained from $C_{\iota}^{\alpha_0}(y)$ by replacing each $y^{r_{\lambda}}$ by 0. We next define

$$W_{\iota}(y) = G_l V_{\iota}^l(y), \quad (l = 0, \dots, n - m),$$

where the constants G_l are defined by

$$(13:5) \quad G_l = (n - l - 1)!/n!.$$

We now assert that the functions $E^{\kappa}(y)$ defined by

$$E^{\kappa}(y) = \int_0^{y^k} W_{k\kappa}(y) dy^k$$

are solutions of (13:4). To prove this we first remark that we may suppose that the components of ι in (13:4) are distinct. Since W_{ι} is skew in ι we then find that

$$(-1)^{c-1} \partial E^{\iota'_c} / \partial y^{i_c} = m W_{\iota} + \int_0^{y^k} G_l \partial V_{\iota_c}^l / \partial y^{i_c} dy^k,$$

where $\iota'_c = (i_1, \dots, i_{c-1}, i_{c+1}, \dots, i_m)$, $\iota_c = (i_1, \dots, i_{c-1}, k, i_{c+1}, \dots, i_m)$ and the sum on the right is for all k not components of ι . From the definition of V_{ι}^l and equation (13:2) we have for fixed k ,

$$\begin{aligned} \int_0^{y^k} \partial V_{\iota_c}^l / \partial y^{i_c} dy^k &= \sum \int_0^{y^k} \partial C_{\iota_c}^{\alpha_0}(y^{\rho}) / \partial y^{i_c} dy^k \\ &= \sum \int_0^{y^k} \partial C_{\iota}^{\alpha_0}(y^{\rho}) / \partial y^k dy^k, \end{aligned}$$

where the second sum is taken for all ρ such that no r_{λ} is a component of ι or is equal to k . On summing for k not a component of ι we have

$$\begin{aligned} \int_0^{y^k} \partial V_{\iota_c}^l / \partial y^{i_c} dy^k &= (n - m - l) \sum C_{\iota}^{\alpha_0}(y^{\rho}) - \sum C_{\iota}^{\alpha_0}(y^{\rho k}) \\ &= (n - m - l) V_{\iota}^l - V_{\iota}^{l+1}, \end{aligned}$$

if we agree that $V_{\iota}^{n-m+1} = 0$. It then follows that the left hand side of (13:4) has the value

$$n G_0 C_{\iota}^{\alpha_0}(y) + V^l (n G_l - l G_l - G_{l-1}) = C_{\iota}^{\alpha_0}(y) \quad (l = 1, \dots, n - m),$$

since we see from (13:5) that $G_0 = 1/n$, $n G_l - l G_l - G_{l-1} = 0$.

LEMMA 13:2. If $H(y, p)$ has the form of Theorem 13:1, then there exist functions H^c such that

$$(13:6) \quad \partial H^c / \partial t_c = H.$$

We define H^c by the equations

$$(13:7) \quad H^c = m!(-1)^{c-1} E^\kappa p_{\beta_c}^\kappa,$$

where the E^κ are the functions obtained in the previous lemma and $\beta_c = (1, \dots, c-1, c+1, \dots, m)$. Then we have

$$\partial H^c / \partial t_c = m!(-1)^{c-1} [\partial E^\kappa / \partial y^d p_{c\beta_c}^{d\kappa} + E^\kappa \partial p_{\beta_c}^\kappa / \partial t_c].$$

Since E^κ is skew in κ , we can write

$$m!(-1)^{c-1} E^\kappa \partial p_{\beta_c}^\kappa / \partial t_c = m E^\kappa \partial S_c^\kappa / \partial t_c,$$

where $S_c^\kappa = (-1)^{c-1} p_c^\kappa D_{\beta_c}^\kappa$ with $\epsilon = (e_1, \dots, e_{m-1})$ is the cofactor of $\partial / \partial t_c$ in the determinant

$$\partial S_c^\kappa / \partial t_c = \begin{vmatrix} \partial / \partial t_c \\ p_c^{\kappa_\mu} \end{vmatrix} \quad (\mu = 1, \dots, m-1).$$

By a lemma of Landers [IX, p. 22], this determinant is zero. Consequently we have

$$\partial H^c / \partial t_c = m!(-1)^{c-1} \partial E^{\kappa'} / \partial y^{i_c} p_{\alpha_0}^{\iota_{\alpha_0}} = m! C_{\alpha_0}^{\alpha_0} p_{\alpha_0}^{\iota_{\alpha_0}} = p_{\alpha_0}^{\iota_{\alpha_0}} C_{\alpha_0}^{\alpha_0} = H,$$

as desired.

To finish the proof of the theorem, let us observe that by Green's lemma, we can write

$$(13:8) \quad I^*(V) = \int_B H^c m_c dB,$$

where m_c are the direction cosines of the outer normal to B . In terms of the quantities S_c^κ introduced in the proof of Lemma 13:2, we have

$$(13:9) \quad H^c m_c = m E^\kappa S_c^\kappa m_c = m E^\kappa \begin{vmatrix} m_c \\ p_c^{\kappa_\mu} \end{vmatrix}.$$

If V and V_0 are two varieties with the same boundary and admitting representation with the same parameters, then we have in a neighborhood of each point of the boundary B of T near which B is represented by functions $t_a(x_1, \dots, x_{m-1})$ the equations $y^i(t(x)) = Y^i(t(x))$, where $y^i(t)$ and $Y^i(t)$ are the functions defining V and V_0 respectively. Differentiation of this relation shows that $(p_c^i - P_c^i) t_\mu^c = 0$, where $P_c^i = \partial Y^i / \partial t_c$ and $t_\mu^c = \partial t_c / \partial x_\mu$. There therefore exist quantities K_i such that $P_c^i = p_c^i + K_i m_c$. Then by elementary determinant theory, we conclude that the determinant in (13:9) is unaltered if we replace

$p_c^{k_\mu}$ by $P_c^{k_\mu}$. It then follows from (13:8) and (13:9) that $I^*(V)$ is equal to $I^*(V_0)$, completing the proof of the theorem.

14. The set $H(\beta; \epsilon)$

In the next section we shall define the constants $M_{\alpha; \gamma}^{\beta; \epsilon}$, originally defined in an inductive manner by (4:5), (4:6) and (4:7), explicitly in terms of a certain subset $H(\beta; \epsilon)$ of the symmetric group $G(\beta\epsilon)$ of all permutations on the components of β and ϵ . Here we shall define $H(\beta; \epsilon)$ and state without proof those properties needed for the subsequent discussion.

We recall that every permutation S in $G(\beta\epsilon)$ is expressible uniquely as the product of commutative cycles on different letters. With this fact in mind we define the set $H(\beta; \epsilon)$ to consist of all of the permutations S in $G(\beta\epsilon)$ which, when expressed as products of cycles on different letters, have no cycle containing more than one component of ϵ . For example, if $g = 1$, so that $\beta = (b)$, we find that $H(\beta; \epsilon)$ consists of the identity permutation and the h transpositions (b, c_μ) ($\mu = 1, \dots, h$). The structure of the set $H(\beta; \epsilon)$ is described in the following

LEMMA 14:1. *If $h = 1$, we have $H(\beta; \epsilon) = G(\beta\epsilon)$. If $g = 1$, then $H(b; \epsilon\epsilon) = H(b; \epsilon) + (b, e)$, the “+” denoting logical sum. Every permutation S in $H(\beta b; \epsilon)$ is expressible uniquely in the form RQ , where R is in $H(b; \epsilon)$ and Q is in $H(\beta; b\epsilon)$; conversely, every such product is in $H(\beta b; \epsilon)$. The set $H(\beta; \epsilon)$ contains $(g + h)!/h!$ permutations.*

15. The explicit definition of $M_{\alpha; \gamma}^{\beta; \epsilon}$

The constants $M_{\alpha; \gamma}^{\beta; \epsilon}$ may now be defined by the equation

$$(15:1) \quad M_{\alpha; \gamma}^{\beta; \epsilon} = \sum (-1)^S \delta_{\alpha \gamma}^{(\beta \epsilon) S}$$

where the sum is taken for all permutations S in $H(\beta; \epsilon)$. Then we find from equation (4:2) and the first sentence in Lemma 14:1 that

$$M_{\alpha; c}^{\beta; e} = D_{\alpha c}^{\beta e}.$$

Setting $g = 1$ in this equation we see that the constants $M_{\alpha; \gamma}^{\beta; \epsilon}$ defined explicitly by (15:1) satisfy (4:5). With the help of the second sentence in Lemma 14:1, we see that

$$M_{\alpha; \gamma c}^{\beta; \epsilon \epsilon} = \sum (-1)^S \delta_{\alpha \gamma c}^{(\beta \epsilon \epsilon) S} - \delta_{\alpha \gamma c}^{\epsilon \epsilon b},$$

where the sum is taken for all permutations S in $H(b; \epsilon)$. Consequently equation (4:6) also holds. To prove (4:7) we perform the following calculation:

$$\begin{aligned} M_{\alpha; s \sigma}^{\beta; b \epsilon} M_{\alpha; \gamma}^{s; \sigma} &= \sum (-1)^Q (-1)^R \delta_{\alpha s \sigma}^{(\beta b \epsilon) Q} \delta_{\alpha \gamma}^{(s \sigma) R} \\ &= \sum (-1)^{RQ} \delta_{\alpha a \gamma}^{(\beta b \epsilon) RQ}, \end{aligned}$$

where the sums are taken for all R in $H(s; \sigma)$ and $H(b; \epsilon)$, respectively, and for all Q in $H(\beta; b\epsilon)$. The third sentence in Lemma 14:1 now shows that (4:7) holds. Thus we have proved that the constants $M_{\alpha; \gamma}^{\beta; \epsilon}$ defined explicitly by (15:1) satisfy (4:5), (4:6) and (4:7), and consequently (5:4) also.

We shall now prove the identity (6:12) which we have used several times. We have from (15:1) and the obvious fact that $H^{-1}(\beta; \epsilon) = H(\beta; \epsilon)$ that

$$D_{\rho\sigma}^{\beta\epsilon} M_{\alpha;\gamma}^{\rho;\sigma} = \sum (-1)^S D_{\rho\sigma}^{\beta\epsilon} \delta_{(\alpha\gamma)S}^{\rho\sigma} = \sum (-1)^S D_{(\alpha\gamma)S}^{\beta\epsilon} = \sum D_{\alpha\gamma}^{\beta\epsilon}.$$

By the last sentence of Lemma 14:1, there are $(g + h)!/h!$ permutations in the set $H(\beta; \epsilon)$. Consequently (6:12) is true.

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